Note on the Analogy Between Momentum Transfer and Heat or Mass Transfer for a Homogeneous Isotropic Turbulent Field

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One of the chief difficulties in the description of turbulent transport is that all real turbulent fields are nonhomogeneous; the properties of the turbulence vary with position in the field.

The examination of the effect of these nonhomogeneities on the transport process has been conditioned by knowledge that has been gained concerning the description of molecular transport. Equations of the same form as Fick's Law have been used to describe turbulent transport of heat or mass:

$$q = \left(\frac{k}{\rho C_p} + \epsilon_c\right) \frac{d(\rho C_p T)}{dX} \qquad (1)$$

One might think that a variation in ϵ_c reflects entirely the nonhomogeneities in the turbulence. This need not be so; as a matter of fact, a homogeneous isotropic turbulent field can exhibit large variations in the coefficient ϵ_c defined by Equation (1). Therefore this equation is probably not the best basis from which to examine the effect of the turbulence nonhomogeneities on the transport process. A correct description of turbulent diffusion from a point source in a homogeneous isotropic field as been given by G. I. Taylor (*1*). In two previous papers by the author (*2*, *3*), Taylor's description has been applied to situations involving wall-transfer processes.

One of the situations considered was the transfer of heat from a hot wall of a channel to the opposite cold wall of a channel. Air was flowing turbulently between the channel walls. Far enough along the channel there was no further change in the temperature of the air, and the temperature profile was fully developed. For a nonturbulent flow in which heat is transferred by molecular conduction a linear temperature profile would be obtained; Equation (1) would suggest the line shown in Figure 1 for a homogeneous turbulent field, if constancy of ϵ_c is assumed to be the criterion for homogeneous turbulence. Experimental data of Page, Corcoran, Schlinger, and Sage (*4*) are also presented in Figure 1. The deviation of the data from a straight line near the wall cannot be attributed entirely to nonhomogeneities. The temperature profile for a homogeneous isotropic turbulence can be calculated by using Taylor's description of point-source diffusion by summing the effects of an infinite number of sources of heat along the hot wall and an infinite number of sinks of heat along the cold wall (*2*). Figure 2 presents the results of such a calculation. A similar deviation from a straight line as is indicated by experimental data is obtained; however the data differ from the calculated profile near the wall, as would be expected, since the assumption of a homogeneous field would not be valid in this region.

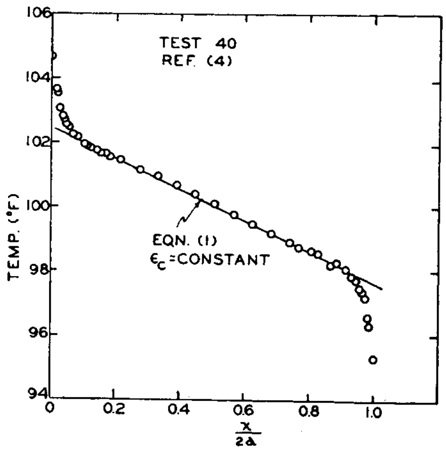

Fig. 1. Data on turbulent heat transfer between two plane walls.

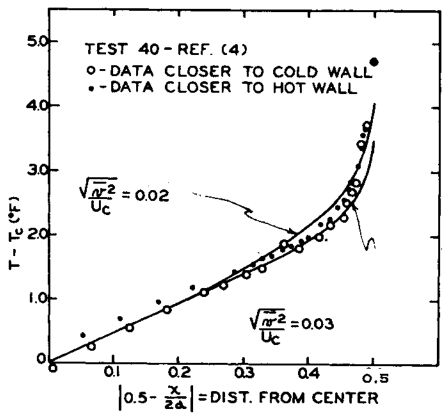

Fig. 2. Heat transfer between two plane walls for a homogeneous isotropic turbulence. $\bar{v}^2\tau$ = 0.005 sq. ft./sec.

The description of turbulent velocity fields has usually been based on an analogy between momentum transfer and mass or heat transfer. Far enough from the entry of a pipe or channel the velocity profile reaches a fully developed condition in which it is not changing. There is no net flow in the radial direction, and momentum transfer is associated with the radial movement of the eddies, as are turbulent heat transfer and mass transfer. An equation of the same form as Equation (1) is often used to describe turbulent momentum transfer:

$$g_c S = \left(\frac{\mu}{\rho} + \epsilon_m\right) \frac{d(\rho U)}{dx} \qquad (2)$$

If ϵ_m is constant, a plot of $U_c - U$ vs. the square of the distance from the channel or pipe center would yield a straight line. On the basis of the calculations on turbulent heat transfer, it is to be expected that variations in ϵ_m or deviations of a plot of $U_c - U$ from linearity should not be due entirely to nonhomogeneities in the turbulence.

In the June, 1958, issue of the *A.I.Ch.E. Journal* (*3*) an expression was derived for the fully developed velocity profile in a pipe by using Taylor's statistical description of turbulent diffusion for a homogeneous isotropic field as the basis of an analogy between momentum transfer and mass or heat transfer. A velocity profile calculated on the basis of this derivation is presented in Figure 3 and compared with data obtained by Deissler (*5*). Details of this calculation are in a Ph.D. thesis by D. L. Flint (*6*). Equations (20) and (43) presented in reference *3* were used for this calculation:

$$U + C_2 = -\int_0^\infty \frac{2S_0}{a} \qquad (3)$$

$$\cdot \sum_{s=1}^{\infty} \exp\left(-\alpha_s^2 K(t - t')\right) \frac{J_0(\alpha_s r)}{J_0(\alpha_s a)}\, dt'$$

$$K(t - t') = \overline{v^2}\tau(t - t') - \overline{v^2}\tau^2 + \overline{v^2}\tau^2 \exp\left(-\frac{(t - t')}{\tau}\right) + \frac{\mu}{\rho}(t - t') \qquad (4)$$

The factor C_2 is a constant of integration. In the paper it was suggested that C_2 be evaluated from the condition that the velocity is zero at the wall. Since this condition is based on experimental observation, and since a homogeneous isotropic field cannot be realized experimentally, it would be fortuitous if this condition were applicable. Therefore in the calculations presented in Figure 3 the constant C_2 is evaluated from the velocity at time zero:

$$C_2 = U_{avg} \qquad (5)$$

The values of the parameters $(\overline{v^2})^{1/2}$ and τ used in the calculations were obtained from measurements of turbulent diffusion of hydrogen from a small tube in a turbulent pipe flow (*6*). The calculated profile does not agree with experimental data. A calculation of the velocity profile for a Reynolds Number of 30,000 was presented in Figure 4 of reference *3*. It has been found that these calculations are in error and that the agreement between the calculated velocity profile and experimental data are much worse than indicated in Figure 4 of reference *3*.* The disagreement could to a large extent be due to a poor choice of the turbulence parameters, $(\overline{v^2})^{1/2}$ and τ.

To examine further the analogy as formulated in reference *3*, calculations have been made of the velocity profile for the conditions prevalent in test 40 of the experiments of Page, Corcoran, Schlinger, and Sage (*4*). The temperature measurements in test 40 are presented in Figure 2 along with calculated profiles based on the development presented in reference *2*

$$T - T_c = \int_0^\infty \frac{q\,dt'}{\sqrt{\pi}\rho C_p[\phi(t - t')]^{1/2}} \cdot \left\{\sum_{n=-\infty}^{n=+\infty} \exp\left(-\frac{(2na + x)^2}{4\phi(t - t')}\right)\right\} \quad (6)$$

However a term for molecular exchange [as is suggested in the footnote on the bottom of page 44 of reference (*2*)] has been included in the expression for $\phi(t - t')$, and $\overline{X^2}_0$ has been dropped from the expression

$$\phi(t - t') = \overline{v^2}\tau(t - t') - \overline{v^2}\tau^2 + \overline{v^2}\tau^2 \cdot \exp\left\{-\frac{(t - t')}{\tau}\right\} + \frac{k}{\rho C_p}(t - t') \quad (7)$$

The two calculated curves used a value of $\overline{v^2}\tau$ equal to an average value of ϵ_c for the data in the center portion of the channel. A value of $(\overline{v^2})^{1/2}/U_c = 0.03$ gives a good fit to the experimental data in the center of the channel, where the turbulence is approximately homogeneous. This value of the turbulence intensity agrees with hot-wire anemometer measurements. The curve based on a value of $(\overline{v^2})^{1/2}/U_c = 0.02$ gives a closer approximation to the wall temperature but a poorer fit to the experimental data in the regions closer to the channel center. From the heat transfer data it appears that values of $\overline{v^2}\tau = 0.005$ sq. ft./sec., and $(\overline{v^2})^{1/2}/U_c = 0.03$ represent the properties of the turbulence in the center of the channel.

An analogy between heat transfer and momentum transfer based on Taylor's statistical description of point-source diffusion may be employed to calculate the velocity profile for this case. The equations for a channel may be developed in the same manner as done for a circular pipe in reference *3*, and the following expression results:

$$U - U_{avg} = \frac{g_c S_0}{\rho a} \cdot \int_0^\infty dt' \left[1 - \frac{a}{\sqrt{\pi}[K(t - t')]^{1/2}} \cdot \sum_{n=-\infty}^{n=+\infty} \exp\left(-\frac{(2na + x)^2}{4K(t - t')}\right)\right] \quad (8)$$

*For this reason much of the discussion in reference *3* pertaining to Figure 4 should be disregarded.

Use of values of $\overline{v^2}\tau = 0.005$ sq. ft./sec. and $(\overline{v^2})^{1/2}/U_c = 0.03$ give a poor fit to the velocity measurements of Page, et al. However values of $\overline{v^2}\tau = 0.80 \times 0.005$ sq. ft./sec. and $(\overline{v^2})^{1/2}/U_c = 0.03$ give a good fit to the experimental data as shown in Figure 4. This value of $\overline{v^2}\tau$

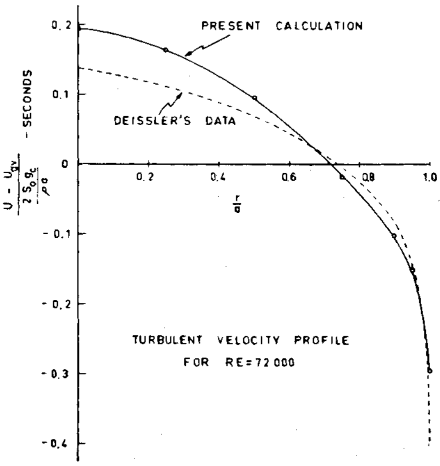

Fig. 3. Velocity profile in a pipe for isotropic homogeneous turbulence.

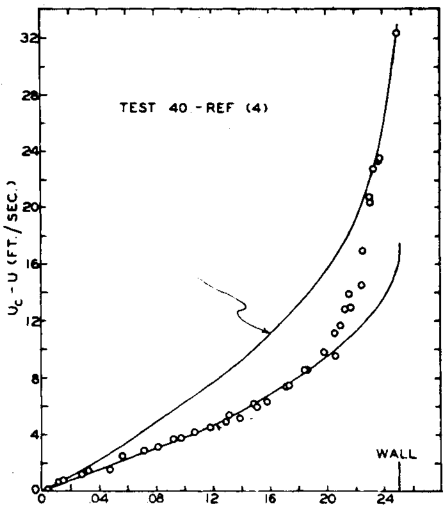

Fig. 4. Velocity profile in a channel for isotropic homogeneous trubulence. $\overline{v^2}\tau$ = 0.004 sq. ft./sec. and $\sqrt{\overline{v^2}}/U_c$ = 0.01 for upper curve. $\overline{v^2}\tau$ = 0.004 sq. ft./sec. and (0.5 − $X/2d$)$\sqrt{\overline{v^2}}/U_c$=0.03 for lower curve.

equals the average value of ϵ_m for the velocity data in the center of the channel. The profile calculated from the analogy by use of these values for the turbulence parameters deviates from a straight line near the wall, as do the experimental data. However the calculated values of $U_c - U$ based on a homogeneous isotropic field are smaller than the measurements near the wall. This would mean that the analogy predicts a finite velocity at the wall for a completely homogeneous field. A velocity profile has also been calculated for $(\overline{v^2})^{1/2}/U_c = 0.01$ and $\overline{v^2}\tau = 0.004$ sq. ft./sec. Although the value of $U_c - U$ at the wall agrees with measurement, the calculated profile gives a poor fit to experimental data in the center of the channel.

From the calculations presented it appears that the analogy as formulated in reference *3* fits velocity data in the center of a channel, and therefore until a more exact theoretical treatment of turbulent velocity fields is available, an analogy based on Taylor's description of point-source diffusion might supply a proper basis from which to examine the effect of nonhomogeneities on the velocity field. However the analogy is open to the same criticism as the analogy based on Equations (1) and (2) in that values of τ different from those obtained from the diffusion of heat or mass must be used to calculate velocity fields.

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NOTATION

a = half width of the channel, pipe radius
C_p = heat capacity
k = thermal conductivity
q = rate of heat transfer per unit area
r = radial distance
S = shear stress at r
S_0 = shear stress at the wall
t = time
t' = time at which diffusing material entered the turbulent field
T = temperature
T_c = temperature at the center of the channel
U = velocity
U_c = velocity at center of the channel or pipe
U_{AV} = average velocity
$(\overline{v^2})^{1/2}$ = root-mean-square turbulent velocity in the x or in the r direction
x = distance from the wall

Greek Letters

α_s = root of $J_0'(\alpha_s a) = 0$
ϵ_c = eddy conductivity analogous to molecular thermal diffusivity
ϵ_m = eddy viscosity analogous to molecular viscosity
μ = molecular viscosity
ρ = density
τ = Lagrangian time scale

LITERATURE CITED

1. Taylor, G. I., *Proc. Roy. Soc.* (*London*), **151A**, 421 (1935).
2. Hanratty, T. J., *A.I.Ch.E. Journal*, **2**, 42 (1956).
3. ———, and D. L. Flint, *ibid.*, **4**, 132 (1958).
4. Page, F., W. H. Corocoran, W. G. Schlinger, and B. H. Sage, *Ind. Eng. Chem.*, **44**, 419 (1952).
5. Deissler, R. G., *Natl. Advisory Comm. Aeronaut., Tech. Note 2138* (1950).
6. Flint, D. L., Ph.D. thesis, Univ. Illinois, Urbana (1958).